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On the bicrossproduct structures for the $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_N}(\mathbf{N}))$ family of algebras

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Abstract. It is shown that the family of deformed algebras $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_N}(N))$ has a different bicrossproduct structure for each $\omega_a = 0$ in analogy to the undeformed case.

1. Introduction

Deformed algebras (usually called ‘quantum groups’) have received much attention since the original works of Drinfel’d, Jimbo and Faddeev, Reshetikhin and Takhtajan [1–4] which gave a (unique) deformation procedure for simple Lie algebras. However, the deformation of non-simple Lie algebras has been characterized by the lack of a definite prescription and this explains why inhomogeneous algebras do not have a unique deformation.

One possible approach to deforming non-simple algebras is by extending the contraction of Lie algebras to the framework of deformed Hopf algebras, an idea originally introduced by Celeghini *et al* [5, 6]. As is well known, the standard İnönü–Wigner [7] contraction of (simple) Lie algebras leads to non-simple algebras which have a semidirect structure, where the ideal is the Abelianized part of the original algebra. By introducing higher powers in the contraction parameter or, equivalently, by performing two (or more) successive contractions it is also possible to arrive to algebras with a central extension structure.

This simple mechanism becomes difficult to implement for deformed algebras for which it is usually necessary to redefine the deformation parameter in terms of the contraction one to have a well-defined contraction limit [5, 6]. This is the case, for instance, of the κ -Poincaré algebra [8, 9] in which the deformation parameter κ appears as a redefinition of the original (adimensional) parameter q of $so_q(3, 2)$ in terms of the de Sitter radius R .

A way to skip some of the problems of the standard contraction procedure for deformed algebras is to use the method of ‘graded’ contractions. This mechanism was put forward by Moody, Montigny and Patera [10, 11] for Lie algebras and has been applied recently to describe a large set of deformed Hopf algebras [12, 13]. The scheme provides the deformation of all motion algebras of flat affine spaces in N dimensions (the deformed Cayley–Klein (CK) algebras $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_N}(N))_{\ddagger}^{\dagger}$) including, the κ -Poincaré algebra in arbitrary dimensions, other deformations of the Poincaré N -dimensional algebra, the Galilei algebra, etc.

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[‡] The orthogonal CK family of algebras are the Lie algebras of the motion groups of real spaces with a projective metric [14, 15].

A different point of view to study inhomogeneous deformed algebras is provided by Majid's *bicrossproduct structure* [16–18]. In this construction we find the analog of the Lie algebra semidirect structure (and of the central extension structure in the more general case) for Hopf algebras and provides, for this reason, an appropriate setting for the study of deformations of inhomogeneous algebras. This structure covers most of the deformed algebras obtained by contraction but not all (see [19]). Thus, in the case of deformed algebras, the correspondence between contraction and semidirect structure that exists in the Lie algebra setting is not straightforward.

Nevertheless, the study of the particular algebras for which the structure of bicrossproduct is present, turns out to be useful to understand its properties because the deformation is mainly encoded in the action and coaction mappings that characterize the bicrossproduct, whereas the (two) Hopf algebras from which the bicrossproduct deformed algebra is constructed are usually undeformed. In the appropriate limit of the deformation parameter we obtain the undeformed algebra, the coaction mapping is trivialized and the action mapping is given by the Lie algebra commutators so that we recover the semidirect product structure. A particular example is the κ -Poincaré algebra [8] the bicrossproduct structure of which was found by Majid and Ruegg [20].

Recently [21] (see also [22]) has been shown that the whole family of deformed inhomogeneous CK algebras $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_N}(N))$ has a bicrossproduct structure, in analogy to the semidirect one that appears after the contraction which goes from $so(p, q)$ to $iso(p, q)$ † and that it remains under all the possible graded contractions. However, the question that naturally arises is whether these contractions carry new bicrossproduct structures related to the semidirect ones of the undeformed algebra which are the result of each contraction (see (2.2) below). We prove in this paper that this is indeed the case so that, for every graded contraction in the inhomogeneous deformed CK family $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_N}(N))$, we have an associated bicrossproduct structure. The (*a priori* non-obvious) fact that all the possible semidirect product structures of the undeformed inhomogeneous CK algebras have a direct counterpart in the deformed case is the main result of this paper.

The paper is organized as follows. In section 2 we provide an account of the (undeformed) CK algebras and their graded contractions. In section 3 some of the results in [21] are summarized. They will permit us in section 4 to show that for each possible graded contraction in the CK family a new bicrossproduct structure arises. Our results are illustrated at the end with some examples.

2. The orthogonal CK family of algebras

Let us start by recalling the definition of the orthogonal CK family of algebras‡. This family, denoted by $so_{\omega_1, \dots, \omega_N}(N+1)$, is a set of algebras characterized by N real parameters $(\omega_1, \dots, \omega_N)$ (within this family we find, for instance, the original $so(N+1)$ algebra, the N -dimensional Poincaré algebra, the Euclidean algebra, etc). Their members share many properties with the (parent) simple Lie algebra and hence may be called ‘quasisimple’. They correspond exactly to the motion algebras of the geometries of a real space with a projective metric in the CK sense [14, 15] and are therefore called CK orthogonal algebras. Their

† Note that associated to each contraction there exists, at the undeformed level, a semidirect product structure and, at the deformed level, a possible bicrossproduct structure associated to it. In this particular case the contraction $so(p, q) \rightarrow iso(p, q)$ gives rise to a semidirect structure in which we have $p+q$ Abelian (momentum) generators and a (pseudo-)orthogonal group acting on them.

‡ These algebras are a subset of all $\mathbb{Z}_2^{\otimes N}$ -graded contractions obtained from $so(N+1)$ depending on $2^N - 1$ real parameters (see [23]).

non-zero brackets are

$$[\mathbb{J}_{ab}, \mathbb{J}_{ac}] = \omega_{ab}\mathbb{J}_{bc} \quad [\mathbb{J}_{ab}, \mathbb{J}_{bc}] = -\mathbb{J}_{ac} \quad [\mathbb{J}_{ac}, \mathbb{J}_{bc}] = \omega_{bc}\mathbb{J}_{ab} \quad (2.1)$$

where $\omega_{ab} = \prod_{s=a+1}^b \omega_s$ and $a < b < c$. By simple rescaling of the generators the values ω_i may be brought to one of the values 1, 0 or -1 .

The structure of these algebras may be defined by two main statements.

- When all ω_i are non-zero the algebra is isomorphic to a certain (pseudo-)orthogonal algebra.

- When a constant $\omega_a = 0$ the resulting algebra $so_{\omega_1, \dots, \omega_a=0, \dots, \omega_N}(N + 1)$ has the semidirect structure

$$so_{\omega_1, \dots, \omega_a=0, \dots, \omega_N}(N + 1) \equiv t \odot (so_{\omega_1, \dots, \omega_{a-1}}(a) \oplus so_{\omega_{a+1}, \dots, \omega_N}(N + 1 - a)) \quad (2.2)$$

where t is an Abelian subalgebra of dimension $\dim t = a(N + 1 - a)$ and the remaining subalgebra is a direct sum. In particular, when $a = 1$ we obtain the usual (pseudo-)orthogonal inhomogeneous algebras $so_{\omega_1=0, \omega_2, \dots, \omega_N}(N + 1)$ with semidirect structure

$$so_{\omega_1=0, \omega_2, \dots, \omega_N}(N + 1) \equiv iso_{\omega_2, \dots, \omega_N}(N) = t_N \odot so_{\omega_2, \dots, \omega_N}(N). \quad (2.3)$$

The structure behind the decomposition (2.2) can be described visually by setting the generators in a triangular array (see figure 1). The generators spanning the subspace t are those inside the rectangle, while the subalgebras $so_{\omega_1, \dots, \omega_{a-1}}(a)$ and $so_{\omega_{a+1}, \dots, \omega_N}(N + 1 - a)$ correspond to the two triangles to the left and below the rectangle respectively. In the $\omega_1 = 0$ ($\omega_N = 0$) case the box is reduced to a single row (column) in the large triangle.

To distinguish between the generators we shall denote by \mathbb{X} those inside the box (Abelian algebra) and by \mathbb{J} those in the two triangles. Namely,

$$\begin{aligned} \mathbb{X}_{ij} &\Rightarrow i < a & \text{and} & & j \geq a \\ \mathbb{J}_{ij} &\Rightarrow i \geq a & \text{or} & & j < a. \end{aligned} \quad (2.4)$$

When two constants are set equal to zero (ω_a and ω_b say) we have two different semidirect decompositions (2.2) corresponding to the constant $\omega_a = 0$ or to the constant $\omega_b = 0$. For instance, the (3, 1)-Galilei algebra appears in this context for $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 1$, $\omega_4 = 1$ and accordingly has two different semidirect structures which correspond to the constants ω_1 and ω_2 .

3. The deformed family of inhomogeneous CK algebras

We summarize in this section some facts concerning the family of deformed inhomogeneous CK algebras. There exists [12, 13] a family of Hopf algebras, denoted by $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_N}(N))$,

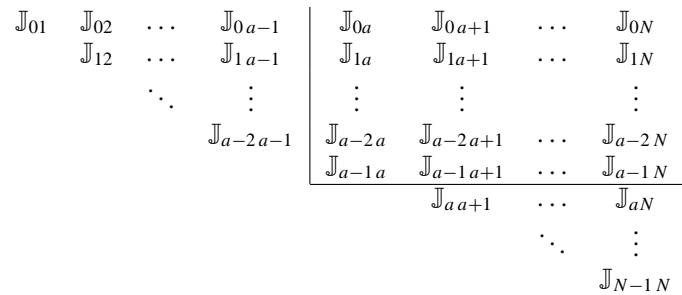


Figure 1. Generators of the CK $so_{\omega_1, \dots, \omega_N}(N + 1)$ algebra.

whose members are a deformation of the CK algebras and, therefore, may be called ‘quantum’ inhomogeneous CK algebras. In [21] it was shown that all these deformed algebras are endowed with a bicrossproduct structure that corresponds to the undeformed semidirect one (2.3) in which the Abelian algebra is given by the single row with generators \mathbb{J}_{0i} (see figure 1 for $a = 1$).

Explicitly the deformed Hopf algebra $\mathcal{U}_\lambda(\mathfrak{iso}_{\omega_2, \dots, \omega_N}(N))$ is given (in the basis in which its bicrossproduct structure is displayed) by

• Commutators

$$\begin{aligned} [\mathbb{J}_{0i}, \mathbb{J}_{0j}] &= 0 & [\mathbb{J}_{0i}, \mathbb{J}_{0N}] &= 0 \\ [\mathbb{J}_{ij}, \mathbb{J}_{ik}] &= \omega_{ij}\mathbb{J}_{jk} & [\mathbb{J}_{ij}, \mathbb{J}_{jk}] &= -\mathbb{J}_{ik} & [\mathbb{J}_{ik}, \mathbb{J}_{jk}] &= \omega_{jk}\mathbb{J}_{ij} \\ [\mathbb{J}_{ij}, \mathbb{J}_{iN}] &= \omega_{ij}\mathbb{J}_{jN} & [\mathbb{J}_{ij}, \mathbb{J}_{jN}] &= -\mathbb{J}_{iN} & [\mathbb{J}_{iN}, \mathbb{J}_{jN}] &= \omega_{jN}\mathbb{J}_{ij} \\ [\mathbb{J}_{ij}, \mathbb{J}_{0k}] &= \delta_{ik}\mathbb{J}_{0j} - \delta_{jk}\omega_{ij}\mathbb{J}_{0i} & [\mathbb{J}_{ij}, \mathbb{J}_{0N}] &= 0 \\ [\mathbb{J}_{iN}, \mathbb{J}_{0j}] &= \delta_{ij} \left(\frac{1 - e^{-2\lambda\mathbb{J}_{0N}}}{2\lambda} - \frac{\lambda}{2} \sum_{s=1}^{N-1} \omega_{sN}\mathbb{J}_{0s}^2 \right) + \lambda\omega_{iN}\mathbb{J}_{0i}\mathbb{J}_{0j} \\ [\mathbb{J}_{iN}, \mathbb{J}_{0N}] &= -\omega_{iN}\mathbb{J}_{0i}. \end{aligned} \quad (3.1)$$

• Coproduct

$$\begin{aligned} \Delta(\mathbb{J}_{0i}) &= e^{-\lambda\mathbb{J}_{0N}} \otimes \mathbb{J}_{0i} + \mathbb{J}_{0i} \otimes 1 & \Delta(\mathbb{J}_{0N}) &= 1 \otimes \mathbb{J}_{0N} + \mathbb{J}_{0N} \otimes 1 \\ \Delta(\mathbb{J}_{ij}) &= 1 \otimes \mathbb{J}_{ij} + \mathbb{J}_{ij} \otimes 1 \\ \Delta(\mathbb{J}_{iN}) &= e^{-\lambda\mathbb{J}_{0N}} \otimes \mathbb{J}_{iN} + \mathbb{J}_{iN} \otimes 1 + \lambda \sum_{s=1}^{i-1} \omega_{iN}\mathbb{J}_{0s} \otimes \mathbb{J}_{si} - \lambda \sum_{s=i+1}^{N-1} \omega_{sN}\mathbb{J}_{0s} \otimes \mathbb{J}_{is}. \end{aligned} \quad (3.2)$$

• Counit

$$\varepsilon(\mathbb{J}_{0i}) = \varepsilon(\mathbb{J}_{0N}) = \varepsilon(\mathbb{J}_{ij}) = \varepsilon(\mathbb{J}_{iN}) = 0. \quad (3.3)$$

• Antipode

$$\begin{aligned} \gamma(\mathbb{J}_{0i}) &= -e^{\lambda\mathbb{J}_{0N}}\mathbb{J}_{0i} & \gamma(\mathbb{J}_{0N}) &= -\mathbb{J}_{0N} & \gamma(\mathbb{J}_{ij}) &= -\mathbb{J}_{ij} \\ \gamma(\mathbb{J}_{iN}) &= -e^{\lambda\mathbb{J}_{0N}}\mathbb{J}_{iN} + \lambda e^{\lambda\mathbb{J}_{0N}} \sum_{s=1}^{i-1} \omega_{iN}\mathbb{J}_{0s}\mathbb{J}_{si} - \lambda e^{\lambda\mathbb{J}_{0N}} \sum_{s=i+1}^{N-1} \omega_{sN}\mathbb{J}_{0s}\mathbb{J}_{is}. \end{aligned} \quad (3.4)$$

4. Bicrossproduct structures for the $\mathcal{U}_\lambda(\mathfrak{iso}_{\omega_2, \dots, \omega_N}(N))$ family of algebras

As a previous result, let us recall the following.

Theorem 4.1 ([21]). The deformed Hopf CK family of algebras $\mathcal{U}_\lambda(\mathfrak{iso}_{\omega_2, \dots, \omega_N}(N))$ has a bicrossproduct structure

$$\mathcal{U}_\lambda(\mathfrak{iso}_{\omega_2, \dots, \omega_N}(N)) = \mathcal{U}(\mathfrak{so}_{\omega_2, \dots, \omega_N}(N))^\beta \triangleright_\alpha \mathcal{U}_\lambda(T_N) \quad (4.1)$$

relative to the right action

$$\alpha(\mathbb{J}_{0i}, \mathbb{J}_{jk}) \equiv \mathbb{J}_{0i} \triangleleft \mathbb{J}_{jk} := [\mathbb{J}_{0i}, \mathbb{J}_{jk}] \quad (4.2)$$

and left coaction β

$$\begin{aligned} \beta(\mathbb{J}_{ij}) &= 1 \otimes \mathbb{J}_{ij} \\ \beta(\mathbb{J}_{iN}) &:= e^{-\lambda\mathbb{J}_{0N}} \otimes \mathbb{J}_{iN} + \lambda \sum_{s=1}^{i-1} \omega_{iN}\mathbb{J}_{0s} \otimes \mathbb{J}_{si} - \lambda \sum_{s=i+1}^{N-1} \omega_{sN}\mathbb{J}_{0s} \otimes \mathbb{J}_{is} \end{aligned} \quad (4.3)$$

where $\mathcal{U}_\lambda(T_N)$ is the Abelian Hopf algebra generated by \mathbb{J}_{0i} and $\mathcal{U}(so_{\omega_2, \dots, \omega_N}(N))$ is the undeformed cocommutative Hopf algebra (with primitive coproduct) generated by \mathbb{J}_{ij} with the commutation relations given in the second and third line of (3.1).

Let us now set $\omega_a = 0$; then the algebra $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_a=0, \dots, \omega_N}(N))$ is given (with the notation in (2.4)) by

• Commutators

$$\mathbb{X}\text{-sector}\{[\mathbb{X}_{ij}, \mathbb{X}_{kl}] = 0 \tag{4.4}$$

$$\mathbb{J}\text{-sector} \left\{ \begin{aligned} &[\mathbb{J}_{0i}, \mathbb{J}_{0k}] = 0 \\ &[\mathbb{J}_{ij}, \mathbb{J}_{0k}] = \delta_{ik}\mathbb{J}_{0j} - \delta_{jk}\omega_{ij}\mathbb{J}_{0i} \\ &[\mathbb{J}_{iN}, \mathbb{J}_{0j}] = \lambda\omega_{iN}\mathbb{X}_{0i}\mathbb{J}_{0j} \\ &[\mathbb{J}_{ij}, \mathbb{J}_{ik}] = \omega_{ij}\mathbb{J}_{jk} \quad [\mathbb{J}_{ij}, \mathbb{J}_{jk}] = -\mathbb{J}_{ik} \quad [\mathbb{J}_{ik}, \mathbb{J}_{jk}] = \omega_{jk}\mathbb{J}_{ij} \\ &[\mathbb{J}_{ij}, \mathbb{J}_{iN}] = \omega_{ij}\mathbb{J}_{jN} \quad [\mathbb{J}_{ij}, \mathbb{J}_{jN}] = -\mathbb{J}_{iN} \quad [\mathbb{J}_{iN}, \mathbb{J}_{jN}] = \omega_{jN}\mathbb{J}_{ij}. \end{aligned} \right. \tag{4.5}$$

$$\mathbb{J}\mathbb{X}\text{-sector} \left\{ \begin{aligned} &[\mathbb{J}_{0i}, \mathbb{X}_{0j}] = [\mathbb{J}_{0i}, \mathbb{X}_{0N}] = 0 \\ &[\mathbb{J}_{ij}, \mathbb{X}_{ik}] = \omega_{ij}\mathbb{X}_{jk} \quad [\mathbb{J}_{ij}, \mathbb{X}_{jk}] = -\mathbb{X}_{ik} \\ &[\mathbb{J}_{jk}, \mathbb{X}_{ij}] = \mathbb{X}_{ik} \quad [\mathbb{J}_{jk}, \mathbb{X}_{ik}] = -\omega_{jk}\mathbb{X}_{ij} \\ &[\mathbb{J}_{ij}, \mathbb{X}_{iN}] = \omega_{ij}\mathbb{X}_{jN} \quad [\mathbb{J}_{ij}, \mathbb{X}_{jN}] = -\mathbb{X}_{iN} \\ &[\mathbb{J}_{jN}, \mathbb{X}_{ij}] = \mathbb{X}_{iN} \quad [\mathbb{J}_{jN}, \mathbb{X}_{iN}] = -\omega_{jN}\mathbb{X}_{ij} \\ &[\mathbb{J}_{0k}, \mathbb{X}_{ij}] = -\delta_{ik}\mathbb{X}_{0j} \quad [\mathbb{J}_{ij}, \mathbb{X}_{0k}] = \delta_{ik}\mathbb{X}_{0j} - \delta_{jk}\omega_{ij}\mathbb{X}_{0i} \\ &[\mathbb{J}_{ij}, \mathbb{X}_{0N}] = 0 \quad [\mathbb{J}_{iN}, \mathbb{X}_{0N}] = -\omega_{iN}\mathbb{X}_{0i} \\ &[\mathbb{J}_{0i}, \mathbb{X}_{jN}] = -\delta_{ij} \left(\frac{1 - e^{-2\lambda\mathbb{X}_{0N}}}{2\lambda} - \frac{\lambda}{2} \sum_{s=a}^{N-1} \omega_{sN}\mathbb{X}_{0s}^2 \right) \\ &[\mathbb{J}_{iN}, \mathbb{X}_{0j}] = \delta_{ij} \left(\frac{1 - e^{-2\lambda\mathbb{X}_{0N}}}{2\lambda} - \frac{\lambda}{2} \sum_{s=a}^{N-1} \omega_{sN}\mathbb{X}_{0s}^2 \right) + \lambda\omega_{iN}\mathbb{X}_{0i}\mathbb{X}_{0j}. \end{aligned} \right. \tag{4.6}$$

• Coproduct

$$\mathbb{X}\text{-sector} \left\{ \begin{aligned} &\Delta\mathbb{X}_{0N} = 1 \otimes \mathbb{X}_{0N} + \mathbb{X}_{0N} \otimes 1 \quad \Delta\mathbb{X}_{ij} = 1 \otimes \mathbb{X}_{ij} + \mathbb{X}_{ij} \otimes 1 \\ &\Delta\mathbb{X}_{0i} = e^{-\lambda\mathbb{X}_{0N}} \otimes \mathbb{X}_{0i} + \mathbb{X}_{0i} \otimes 1 \\ &\Delta\mathbb{X}_{iN} = e^{-\lambda\mathbb{X}_{0N}} \otimes \mathbb{X}_{iN} + \mathbb{X}_{iN} \otimes 1 - \lambda \sum_{s=a}^{N-1} \omega_{sN}\mathbb{X}_{0s} \otimes \mathbb{X}_{is} \end{aligned} \right. \tag{4.7}$$

$$\mathbb{J}\text{-sector} \left\{ \begin{aligned} &\Delta\mathbb{J}_{0i} = e^{-\lambda\mathbb{X}_{0N}} \otimes \mathbb{J}_{0i} + \mathbb{J}_{0i} \otimes 1 \quad \Delta\mathbb{J}_{ij} = 1 \otimes \mathbb{J}_{ij} + \mathbb{J}_{ij} \otimes 1 \\ &\Delta\mathbb{J}_{iN} = e^{-\lambda\mathbb{X}_{0N}} \otimes \mathbb{J}_{iN} + \mathbb{J}_{iN} \otimes 1 + \lambda \sum_{s=1}^{a-1} \omega_{iN}\mathbb{J}_{0s} \otimes \mathbb{X}_{si} \\ &\quad + \lambda \sum_{s=a}^{i-1} \omega_{iN}\mathbb{X}_{0s} \otimes \mathbb{J}_{si} - \lambda \sum_{s=i+1}^{N-1} \omega_{sN}\mathbb{X}_{0s} \otimes \mathbb{J}_{is}. \end{aligned} \right. \tag{4.8}$$

• Counit

$$\begin{aligned} \varepsilon(\mathbb{J}_{ij}) &= \varepsilon(\mathbb{J}_{0i}) = \varepsilon(\mathbb{J}_{iN}) = 0 \\ \varepsilon(\mathbb{X}_{ij}) &= \varepsilon(\mathbb{X}_{0i}) = \varepsilon(\mathbb{X}_{0N}) = \varepsilon(\mathbb{X}_{iN}) = 0. \end{aligned} \tag{4.9}$$

• Antipode

$$\mathbb{X}\text{-sector} \begin{cases} \gamma(\mathbb{X}_{0N}) = -\mathbb{X}_{0N} & \gamma(\mathbb{X}_{ij}) = -\mathbb{X}_{ij} \\ \gamma(\mathbb{X}_{0i}) = -e^{\lambda\mathbb{X}_{0N}} \mathbb{X}_{0i} \\ \gamma(\mathbb{X}_{iN}) = -e^{\lambda\mathbb{X}_{0N}} \mathbb{X}_{iN} - \lambda e^{\lambda\mathbb{X}_{0N}} \sum_{s=a}^{N-1} \omega_{sN} \mathbb{X}_{0s} \mathbb{X}_{is} \end{cases} \quad (4.10)$$

$$\mathbb{J}\text{-sector} \begin{cases} \gamma(\mathbb{J}_{ij}) = -\mathbb{J}_{ij} \\ \gamma(\mathbb{J}_{0i}) = -e^{\lambda\mathbb{X}_{0N}} \mathbb{J}_{0i} \\ \gamma(\mathbb{J}_{iN}) = -e^{\lambda\mathbb{X}_{0N}} \mathbb{J}_{iN} + \lambda e^{\lambda\mathbb{X}_{0N}} \sum_{s=1}^{a-1} \omega_{iN} \mathbb{J}_{0s} \mathbb{X}_{si} \\ \quad + \lambda e^{\lambda\mathbb{X}_{0N}} \sum_{s=a}^{i-1} \omega_{iN} \mathbb{X}_{0s} \mathbb{J}_{si} - \lambda e^{\lambda\mathbb{X}_{0N}} \sum_{s=i+1}^{N-1} \mathbb{X}_{0s} \mathbb{J}_{is}. \end{cases} \quad (4.11)$$

This algebra, as a result of theorem 4.1, has a bicrossproduct structure (4.1). However, the algebra (4.4)–(4.11) does not present directly a bicrossproduct structure for the decomposition (2.2). This is due to the term $\lambda \sum_{s=1}^{a-1} \omega_{iN} \mathbb{J}_{0s} \otimes \mathbb{X}_{si}$ in the \mathbb{J}_{iN} coproduct (second line in (4.8)) and to the commutator $[\mathbb{J}_{iN}, \mathbb{J}_{0j}]$ (third line in (4.5)) that does not close a \mathbb{J} algebra[†].

Let us define

$$\hat{J}_{iN} = \lambda \sum_{s=1}^{a-1} \omega_{iN} \mathbb{J}_{0s} \mathbb{X}_{si} \quad (4.12)$$

that verifies

$$\Delta \hat{J}_{iN} = e^{-\lambda\mathbb{X}_{0N}} \otimes \hat{J}_{iN} + \hat{J}_{iN} \otimes 1 + \lambda \sum_{s=1}^{a-1} \omega_{iN} \mathbb{J}_{0s} \otimes \mathbb{X}_{si} + \lambda \sum_{s=1}^{a-1} \omega_{iN} e^{-\lambda\mathbb{X}_{0N}} \mathbb{X}_{si} \otimes \mathbb{J}_{0s} \quad (4.13)$$

and

$$[\hat{J}_{iN}, \mathbb{J}_{0j}] = \lambda \omega_{iN} \mathbb{J}_{0j} \mathbb{X}_{0i}. \quad (4.14)$$

Thus, the change of basis $\mathbb{J}_{iN} \rightarrow \mathbb{J}_{iN} - \hat{J}_{iN}$ solves the two difficulties pointed out before.

Specifically, if we introduce the new set of generators

$$\begin{aligned} J_{0i} &= \mathbb{J}_{0i} & X_{0i} &= \mathbb{X}_{0i} & X_{0N} &= \mathbb{X}_{0N} \\ J_{ij} &= \mathbb{J}_{ij} & X_{ij} &= \mathbb{X}_{ij} & X_{iN} &= \mathbb{X}_{iN} \\ J_{iN} &= \mathbb{J}_{iN} - \hat{J}_{iN} \end{aligned} \quad (4.15)$$

the algebra $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_a=0, \dots, \omega_N}(N))$ is written as

• Commutators

$$X\text{-sector} \begin{cases} [X_{0i}, X_{0j}] = [X_{0i}, X_{0N}] = 0 \\ [X_{ij}, X_{0k}] = [X_{ij}, X_{kl}] = [X_{ij}, X_{kN}] = 0 \\ [X_{iN}, X_{0j}] = [X_{iN}, X_{0N}] = [X_{iN}, X_{jk}] = [X_{iN}, X_{jN}] = 0 \end{cases} \quad (4.16)$$

[†] Nevertheless in the particular case $a = N$ ($\omega_N = 0$) these terms are not present, and the change of basis given in (4.15) is not necessary (note that for $\omega_N = 0$ the change of basis is trivial).

$$J\text{-sector} \left\{ \begin{array}{l} [J_{0i}, J_{0k}] = 0 \\ [J_{ij}, J_{0k}] = \delta_{ik} J_{0j} - \delta_{jk} \omega_{ij} J_{0i} \\ [J_{iN}, J_{0j}] = 0 \\ [J_{ij}, J_{ik}] = \omega_{ij} J_{jk} \quad [J_{ij}, J_{jk}] = -J_{ik} \quad [J_{ik}, J_{jk}] = \omega_{jk} J_{ij} \\ [J_{ij}, J_{iN}] = \omega_{ij} J_{jN} \quad [J_{ij}, J_{jN}] = -J_{iN} \quad [J_{iN}, J_{jN}] = \omega_{jN} J_{ij} \end{array} \right. \quad (4.17)$$

$$JX\text{-sector} \left\{ \begin{array}{l} [J_{0i}, X_{0j}] = [J_{0i}, X_{0N}] = 0 \\ [J_{ij}, X_{ik}] = \omega_{ij} X_{jk} \quad [J_{ij}, X_{jk}] = -X_{ik} \\ [J_{jk}, X_{ij}] = X_{ik} \quad [J_{jk}, X_{ik}] = -\omega_{jk} X_{ij} \\ [J_{ij}, X_{iN}] = \omega_{ij} X_{jN} \quad [J_{ij}, X_{jN}] = -X_{iN} \quad [J_{jN}, X_{ij}] = X_{iN} \\ [J_{jN}, X_{iN}] = -\omega_{jN} X_{ij} + \lambda \omega_{jN} \left(\frac{1 - e^{-2\lambda X_{0N}}}{2\lambda} - \frac{\lambda}{2} \sum_{s=a}^{N-1} \omega_{sN} X_{0s}^2 \right) X_{ij} \\ [J_{0k}, X_{ij}] = -\delta_{ik} X_{0j} \quad [J_{ij}, X_{0k}] = \delta_{ik} X_{0j} - \delta_{jk} \omega_{ij} X_{0i} \\ [J_{ij}, X_{0N}] = 0 \quad [J_{iN}, X_{0N}] = -\omega_{iN} X_{0i} \\ [J_{kN}, X_{ij}] = \delta_{jk} X_{iN} + \lambda \omega_{kN} X_{0j} X_{ik} \\ [J_{0i}, X_{jN}] = -\delta_{ij} \left(\frac{1 - e^{-2\lambda X_{0N}}}{2\lambda} - \frac{\lambda}{2} \sum_{s=a}^{N-1} \omega_{sN} X_{0s}^2 \right) \\ [J_{iN}, X_{0j}] = \delta_{ij} \left(\frac{1 - e^{-2\lambda X_{0N}}}{2\lambda} - \frac{\lambda}{2} \sum_{s=a}^{N-1} \omega_{sN} X_{0s}^2 \right) + \lambda \omega_{iN} X_{0i} X_{0j}. \end{array} \right. \quad (4.18)$$

• Coproduct

$$X\text{-sector} \left\{ \begin{array}{l} \Delta X_{0N} = 1 \otimes X_{0N} + X_{0N} \otimes 1 \quad \Delta X_{ij} = 1 \otimes X_{ij} + X_{ij} \otimes 1 \\ \Delta X_{0i} = e^{-\lambda X_{0N}} \otimes X_{0i} + X_{0i} \otimes 1 \\ \Delta X_{iN} = e^{-\lambda X_{0N}} \otimes X_{iN} + X_{iN} \otimes 1 - \lambda \sum_{s=a}^{N-1} \omega_{sN} X_{0s} \otimes X_{is} \end{array} \right. \quad (4.19)$$

$$J\text{-sector} \left\{ \begin{array}{l} \Delta J_{0i} = e^{-\lambda X_{0N}} \otimes J_{0i} + J_{0i} \otimes 1 \quad \Delta J_{ij} = 1 \otimes J_{ij} + J_{ij} \otimes 1 \\ \Delta J_{iN} = e^{-\lambda X_{0N}} \otimes J_{iN} + J_{iN} \otimes 1 - \lambda \sum_{s=1}^{a-1} \omega_{iN} e^{-\lambda X_{0N}} X_{si} \otimes J_{0s} \\ \quad + \lambda \sum_{s=a}^{i-1} \omega_{iN} X_{0s} \otimes J_{si} - \lambda \sum_{s=i+1}^{N-1} \omega_{sN} X_{0s} \otimes J_{is}. \end{array} \right. \quad (4.20)$$

• Counit

$$\begin{aligned} \varepsilon(J_{ij}) &= \varepsilon(J_{0i}) = \varepsilon(J_{iN}) = 0 \\ \varepsilon(X_{ij}) &= \varepsilon(X_{0i}) = \varepsilon(X_{0N}) = \varepsilon(X_{iN}) = 0. \end{aligned} \quad (4.21)$$

• Antipode

$$X\text{-sector} \left\{ \begin{array}{l} \gamma(X_{0N}) = -X_{0N} \quad \gamma(X_{ij}) = -X_{ij} \\ \gamma(X_{0i}) = -e^{\lambda X_{0N}} X_{0i} \\ \gamma(X_{iN}) = -e^{\lambda X_{0N}} X_{iN} - \lambda e^{\lambda X_{0N}} \sum_{s=a}^{N-1} \omega_{sN} X_{0s} X_{is} \end{array} \right. \quad (4.22)$$

$$J\text{-sector} \begin{cases} \gamma(J_{ij}) = -J_{ij} \\ \gamma(J_{0i}) = -e^{\lambda X_{0N}} J_{0i} \\ \gamma(J_{iN}) = -e^{\lambda X_{0N}} J_{iN} - \lambda e^{\lambda X_{0N}} \sum_{s=1}^{a-1} \omega_{iN} X_{si} J_{0s} \\ \quad + \lambda e^{\lambda X_{0N}} \sum_{s=a}^{i-1} \omega_{iN} X_{0s} J_{si} - \lambda e^{\lambda X_{0N}} \sum_{s=i+1}^{N-1} \omega_{sN} X_{0s} X_{is}. \end{cases} \quad (4.23)$$

In this basis we now state the following.

Theorem 4.2. The algebra $\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_a=0, \dots, \omega_N}(N))$ has the bicrossproduct structure

$$\mathcal{U}_\lambda(iso_{\omega_2, \dots, \omega_a=0, \dots, \omega_N}(N)) = \mathcal{U}(so_{\omega_1=0, \dots, \omega_{a-1}}(a)) \oplus \mathcal{U}(so_{\omega_{a+1}, \dots, \omega_N}(N+1-a))^\beta \triangleright_\alpha \mathcal{U}_\lambda(T_{a(N+1-a)}) \quad (4.24)$$

where $\mathcal{U}(so_{\omega_1=0, \dots, \omega_{a-1}}(a))$ is the undeformed Hopf algebra generated by $\{J_{ij}, i < (a-1), j < a\}$, $\mathcal{U}(so_{\omega_{a+1}, \dots, \omega_N}(N+1-a))$ is spanned by the generators $\{J_{ij}, i > (a-1), j > a\}$ and $\mathcal{U}_\lambda(T_{a(N+1-a)})$ is the deformed Abelian algebra generated by X_{ij} (recall that those generators are restricted to the indices $i < a$ and $j \geq a$, (2.4)). The right action $\alpha : \mathcal{U}_\lambda(T_{a(N+1-a)}) \otimes \mathcal{U}(so_{\omega_1=0, \dots, \omega_{a-1}}(a)) \oplus \mathcal{U}(so_{\omega_{a+1}, \dots, \omega_N}(N+1-a)) \rightarrow \mathcal{U}_\lambda(T_{a(N+1-a)})$ is defined by (4.18) through

$$\alpha(X_{ij}, J_{kl}) \equiv X_{ij} \triangleleft J_{kl} := [X_{ij}, J_{kl}] \quad (4.25)$$

and the (left) coaction $\beta : \mathcal{U}(so_{\omega_1=0, \dots, \omega_{a-1}}(a)) \oplus \mathcal{U}(so_{\omega_{a+1}, \dots, \omega_N}(N+1-a)) \rightarrow \mathcal{U}_\lambda(T_{a(N+1-a)}) \otimes (\mathcal{U}(so_{\omega_1=0, \dots, \omega_{a-1}}(a)) \oplus \mathcal{U}(so_{\omega_{a+1}, \dots, \omega_N}(N+1-a)))$ is designed to reproduce the coproduct (4.20)

$$\begin{aligned} \beta(J_{ij}) &= 1 \otimes J_{ij} \\ \beta(J_{0i}) &= e^{-\lambda X_{0N}} \otimes J_{ij} \\ \beta(J_{iN}) &= e^{-\lambda X_{0N}} \otimes J_{iN} - \lambda \sum_{s=1}^{a-1} \omega_{iN} e^{-\lambda X_{0N}} X_{si} \otimes J_{0s} + \lambda \sum_{s=a}^{i-1} \omega_{iN} X_{0s} \otimes J_{si} \\ &\quad - \lambda \sum_{s=i+1}^{N-1} \omega_{sN} X_{0s} \otimes J_{is}. \end{aligned} \quad (4.26)$$

From theorem 4.2 it is easy to check the following.

Corollary 4.1. Associated to each graded contraction in the inhomogeneous CK family of deformed algebras we have a different bicrossproduct structure related to the corresponding Lie algebra semidirect structure that appears in the contraction (see (2.2)). These bicrossproduct structures are preserved under further (graded) contraction processes.

5. Examples

5.1. $N = 3$ case

In the $N = 3$ case we obtain (in the basis of (3.1)–(3.4)) the following equations

$$\begin{aligned}
 [\mathbb{J}_{0i}, \mathbb{J}_{0j}] &= 0 \quad i, j = 1, 2, 3 \\
 [\mathbb{J}_{12}, \mathbb{J}_{13}] &= \omega_2 \mathbb{J}_{23} \quad [\mathbb{J}_{13}, \mathbb{J}_{23}] = \omega_3 \mathbb{J}_{12} \quad [\mathbb{J}_{12}, \mathbb{J}_{23}] = -\mathbb{J}_{13} \\
 [\mathbb{J}_{12}, \mathbb{J}_{03}] &= 0 \quad [\mathbb{J}_{12}, \mathbb{J}_{01}] = \mathbb{J}_{02} \quad [\mathbb{J}_{12}, \mathbb{J}_{02}] = -\omega_2 \mathbb{J}_{01} \\
 [\mathbb{J}_{13}, \mathbb{J}_{03}] &= -\omega_2 \omega_3 \mathbb{J}_{01} \quad [\mathbb{J}_{23}, \mathbb{J}_{03}] = -\omega_2 \mathbb{J}_{02} \\
 [\mathbb{J}_{13}, \mathbb{J}_{01}] &= \frac{1 - e^{-2\lambda \mathbb{J}_{03}}}{2\lambda} + \frac{\lambda}{2} \omega_3 (\omega_2 \mathbb{J}_{01}^2 - \mathbb{J}_{02}^2) \\
 [\mathbb{J}_{23}, \mathbb{J}_{02}] &= \frac{1 - e^{-2\lambda \mathbb{J}_{03}}}{2\lambda} - \frac{\lambda}{2} \omega_3 (\omega_2 \mathbb{J}_{01}^2 - \mathbb{J}_{02}^2) \\
 [\mathbb{J}_{13}, \mathbb{J}_{02}] &= \lambda \omega_2 \omega_3 \mathbb{J}_{01} \mathbb{J}_{02} \quad [\mathbb{J}_{23}, \mathbb{J}_{01}] = \boxed{\lambda \omega_3 \mathbb{J}_{01} \mathbb{J}_{02}} \\
 \Delta \mathbb{J}_{0i} &= e^{-\lambda \mathbb{J}_{03}} \otimes \mathbb{J}_{0i} + \mathbb{J}_{0i} \otimes 1 \quad i = 1, 2 \\
 \Delta \mathbb{J}_{03} &= 1 \otimes \mathbb{J}_{03} + \mathbb{J}_{03} \otimes 1 \quad \Delta \mathbb{J}_{12} = 1 \otimes \mathbb{J}_{12} + \mathbb{J}_{12} \otimes 1 \\
 \Delta \mathbb{J}_{13} &= e^{-\lambda \mathbb{J}_{03}} \otimes \mathbb{J}_{13} + \mathbb{J}_{13} \otimes 1 - \lambda \omega_3 \mathbb{J}_{02} \otimes \mathbb{J}_{12} \\
 \Delta \mathbb{J}_{23} &= e^{-\lambda \mathbb{J}_{03}} \otimes \mathbb{J}_{23} + \mathbb{J}_{23} \otimes 1 + \boxed{\lambda \omega_3 \mathbb{J}_{01} \otimes \mathbb{J}_{12}}.
 \end{aligned} \tag{5.1}$$

We have stressed with a box the terms that might not allow the algebra to be a bicrossproduct (see section 4). If $\omega_3 = 0$ these terms cancel[†] but if $\omega_2 = 0$ we keep them and we need the change of basis given in (4.15). In the new basis we have

$$\Delta J_{23} = e^{-\lambda X_{03}} \otimes J_{23} + J_{23} \otimes 1 - \lambda \omega_3 X_{12} e^{-\lambda X_{03}} \otimes J_{01} \tag{5.3}$$

and

$$[J_{23}, J_{01}] = 0. \tag{5.4}$$

Thus, the terms marked above have disappeared after the change of basis and we have a bicrossproduct structure as given in theorem 4.2.

5.2. A particular case: the Heisenberg–Weyl algebra

Now we are going to study the case $a = N$ (i.e. $\omega_N = 0$). First, let us rename the generators in the basis (4.15) as

$$J_{0i} = X_i \quad J_{ij} = J_{ij} \quad X_{iN} = Y_i \quad X_{0N} = \Xi \tag{5.5}$$

(note that for $\omega_N = 0$ the X sector is reduced to a single column in the triangular array in figure 1). Now the equations (4.16)–(4.18) acquire the form

$$\begin{aligned}
 [X_i, \Xi] &= [Y_i, \Xi] = [J_{ij}, \Xi] = 0 \\
 [X_i, X_j] &= 0 \quad [Y_i, Y_j] = 0 \\
 [J_{ij}, J_{ik}] &= \omega_{ij} J_{jk} \quad [J_{ij}, J_{jk}] = -J_{ik} \quad [J_{ik}, J_{jk}] = \omega_{jk} J_{ij} \\
 [J_{ij}, X_k] &= \delta_{ik} X_j - \delta_{jk} \omega_{ij} X_i \quad [J_{ij}, Y_k] = \delta_{ik} \omega_{ij} Y_j - \delta_{jk} Y_i \\
 [X_i, Y_j] &= -\delta_{ij} \left(\frac{1 - e^{-2\lambda \Xi}}{2\lambda} \right).
 \end{aligned} \tag{5.6}$$

[†] Note that for $\omega_3 = 0$ (i.e. $\omega_N = 0$) the change of basis (4.15) is trivial (see (4.12) and the previous footnote).

In this way, we easily recognize the deformed Heisenberg–Weyl (HW) algebra [19] where Ξ is the central generator and the J_{ij} generators act as a rotation group on the X_i and Y_i generators. The coproduct (4.19), (4.20) takes the form

$$\begin{aligned}\Delta \Xi &= 1 \otimes \Xi + \Xi \otimes 1 & \Delta J_{ij} &= 1 \otimes J_{ij} + J_{ij} \otimes 1 \\ \Delta X_i &= e^{-\lambda \Xi} \otimes X_i + X_i \otimes 1 & \Delta Y_i &= e^{-\lambda \Xi} \otimes Y_i + Y_i \otimes 1.\end{aligned}\quad (5.7)$$

From the arguments given above we know that this algebra has two different bicrossproduct (semidirect like) structures, one for the Abelian algebra generated by $\{X_i, \Xi\}$, and the other for the Abelian algebra generated by $\{Y_i, \Xi\}$ [19].

However, in this case, we have an additional cocycle bicrossproduct structure (analogue to the undeformed central extension structure of the HW-algebra). To see this let us define the algebra \mathcal{H} as the undeformed algebra generated by $\{X_i, Y_i, J_{ij}\}$ with primitive coproduct and commutators

$$\begin{aligned}[J_{ij}, J_{ik}] &= \omega_{ij} J_{jk} & [J_{ij}, J_{jk}] &= -J_{ik} & [J_{ik}, J_{jk}] &= \omega_{jk} J_{ij} \\ [J_{ij}, X_k] &= \delta_{ik} X_j - \delta_{jk} \omega_{ij} X_i & [J_{ij}, Y_k] &= \delta_{ik} \omega_{ij} Y_j - \delta_{jk} Y_i \\ [X_i, X_j] &= 0 & [Y_i, Y_j] &= 0 & [X_i, Y_j] &= 0\end{aligned}\quad (5.8)$$

(note that all the commutators are identical to those in (5.6) but the $[X_i, Y_j]$ one that now is Abelian). The algebra \mathcal{A} is the undeformed algebra $\mathcal{U}(\Xi)$. Now if we define the right action $\triangleleft : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{A}$

$$\Xi \triangleleft J_{ij} = 0 \quad \Xi \triangleleft X_i = 0 \quad \Xi \triangleleft Y_i = 0 \quad (5.9)$$

(central extension means trivial action), the left coaction $\beta : \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$

$$\beta(J_{ij}) = 1 \otimes J_{ij} \quad \beta(X_i) = e^{-\lambda \Xi} \otimes X_i \quad \beta(Y_i) = e^{-\lambda \Xi} \otimes X_i \quad (5.10)$$

the antisymmetric two-cocycle $\xi : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{A}^\dagger$

$$\xi(X_i, Y_j) = -\xi(Y_j, X_i) = -\frac{\delta_{ij}}{2} \left(\frac{1 - e^{-2\lambda \Xi}}{2\lambda} \right) \quad (5.11)$$

and a trivial ‘two-cocycle’ the HW algebra is given by the bicrossproduct

$$\mathcal{U}_\lambda(HW) = \mathcal{H} \triangleright \triangleleft_{\xi} \mathcal{A}. \quad (5.12)$$

In this form it is easy to recover the dual algebra $\text{Fun}_\lambda(HW)$ [19]. Let R_{ij} be the dual generators corresponding to the undeformed ‘rotation’ algebra generated by J_{ij}^\ddagger and let x_i, y_j be the dual coordinates to the generators X_i, Y_j . Then, the algebra H dual to \mathcal{H} is given by

$$\Delta R_{ij} = R_{ik} \otimes R_{kj} \quad (5.13)$$

$$\begin{aligned}\Delta x_i &= 1 \otimes x_i + x_k \otimes R_{ki} & \Delta y_i &= 1 \otimes y_i + y_k \otimes R_{ik}^{-1} \\ [R_{ij}, R_{kl}] &= [R_{ij}, x_k] = [R_{ij}, y_k] = [x_i, y_j] = 0.\end{aligned}\quad (5.14)$$

If we introduce the coordinate χ dual to the central generator Ξ we may complete the dual algebra by dualizing the left coaction (5.10) and the two-cocycle (5.11). The left action is defined as the dual to the left coaction

$$\chi \triangleright x_i = [\chi, x_i] = -\lambda x_i \quad \chi \triangleright y_i = [\chi, y_i] = -\lambda y_i \quad \chi \triangleright R_{ij} = [\chi, R_{ij}] = 0 \quad (5.15)$$

† The antisymmetric form of the cocycle is a matter of convention; different forms of the cocycle are related by a coboundary change (see [24] for an explicit example).

‡ This algebra is a true rotation algebra for $\omega_i = 1 \ i = 1, \dots, N-1$; in general it is an inhomogeneous algebra (if some $\omega = 0$) or a pseudo-orthogonal algebra. The dual algebra is given by the matrix representation R_{ij} .

and the dual to the two-cocycle defines the two-cocycle§

$$\bar{\psi}(\chi) = \frac{1}{2}(y_i \otimes R_{ji}^{-1}x_j - x_i \otimes R_{ij}y_j). \quad (5.16)$$

Thus, the coproduct is given by

$$\Delta\chi = 1 \otimes \chi + \chi \otimes 1 + \frac{1}{2}(y_i \otimes R_{ji}^{-1}x_j - x_i \otimes R_{ij}y_j). \quad (5.17)$$

As we can see the bicrossproduct structure (with cocycle in this case) allows us to recover $\text{Fun}_\lambda(\text{HW})$ in an easy way from the enveloping (dual) algebra $\mathcal{U}_\lambda(\text{HW})$.

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§ As mentioned earlier we may choose a different form of the two-cocycle. For instance, $\bar{\psi}(\chi) = y_i \otimes R_{ji}^{-1}x_j$ is also a two-cocycle (related to (5.16) by the cocoboundary $\frac{1}{2}y_i x_i$).

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